

On Gerstenhaber's theorem for spaces of nilpotent matrices over a skew field

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Abstract

Let \mathbb{K} be a skew field, and \mathbb{K}_0 be a subfield of the central subfield of \mathbb{K} such that \mathbb{K} has finite dimension q over \mathbb{K}_0 . Let \mathcal{V} be a \mathbb{K}_0 -linear subspace of $n \times n$ nilpotent matrices with entries in \mathbb{K} . We show that the dimension of \mathcal{V} is bounded above by $q \binom{n}{2}$, and that equality occurs if and only if \mathcal{V} is similar to the space of all $n \times n$ strictly upper-triangular matrices over \mathbb{K} . This generalizes famous theorems of Gerstenhaber and Serezhkin, which cover the special case $\mathbb{K} = \mathbb{K}_0$.

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1 Introduction

In this article, we let \mathbb{K} be an arbitrary skew field, and \mathbb{K}_0 be a subfield of the central subfield of \mathbb{K} over which \mathbb{K} has finite dimension q . The set \mathbb{K}^n is always endowed with its canonical structure of right- \mathbb{K} -vector space. We denote by $M_{n,p}(\mathbb{K})$ the set of all $n \times p$ matrices with entries in \mathbb{K} , endowed with its canonical structure of vector space over \mathbb{K}_0 . We set $M_n(\mathbb{K}) := M_{n,n}(\mathbb{K})$, and denote by $GL_n(\mathbb{K})$ its group of invertible elements. We denote by $NT_n(\mathbb{K})$ the set of all strictly upper-triangular matrices of $M_n(\mathbb{K})$.

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The transpose of a matrix M is denoted by M^T , and its trace by $\text{tr}(M)$. The relation of similarity between matrices is denoted by \simeq and is naturally extended to subsets of $M_n(\mathbb{K})$.

A linear subspace \mathcal{V} of $M_n(\mathbb{K})$ (over \mathbb{K}_0) is called *nilpotent* when all its elements are nilpotent matrices. In that case, we note that, for every $P \in \text{GL}_n(\mathbb{K})$, the set $P\mathcal{V}P^{-1}$ is a nilpotent linear subspace of $M_n(\mathbb{K})$ with the same dimension as \mathcal{V} .

In his first entry in a series of four landmark papers [1], Murray Gerstenhaber studied the structure of such nilpotent subspaces. Here is his most famous result:

Theorem 1 (Gerstenhaber, Serezhkin). *Assume that \mathbb{K} is commutative, and let \mathcal{V} be a nilpotent linear subspace of the \mathbb{K} -vector space $M_n(\mathbb{K})$. Then $\dim \mathcal{V} \leq \binom{n}{2}$, and equality occurs if and only if \mathcal{V} is similar to $\text{NT}_n(\mathbb{K})$.*

Our main aim here is to prove the following generalization to skew fields:

Theorem 2. *Let \mathcal{V} be a nilpotent linear subspace of $M_n(\mathbb{K})$ (over \mathbb{K}_0). Then:*

- (a) $\dim \mathcal{V} \leq q \binom{n}{2}$.
- (b) *If $\dim \mathcal{V} = q \binom{n}{2}$, then \mathcal{V} is similar to $\text{NT}_n(\mathbb{K})$.*

If \mathbb{K} is finite (and therefore commutative), choosing \mathbb{K}_0 as its prime subfield yields the following corollary:

Corollary 3. *Assume \mathbb{K} is finite with cardinality p . Let \mathcal{V} be a subgroup of $(M_n(\mathbb{K}), +)$ in which every matrix is nilpotent. Then $\#\mathcal{V} \leq p^{\binom{n}{2}}$, and equality occurs only if \mathcal{V} is similar to $\text{NT}_n(\mathbb{K})$.*

At the time of [1], Gerstenhaber was actually able to prove Theorem 1 only for fields with at least n elements, mostly because his methods relied on the use of polynomials. A lot of progress has been made since then: we now have elementary and elegant proofs of the inequality statement that are valid for every field [3, 2], and the case of equality has been obtained for an arbitrary field by V.N. Serezhkin [7] (for fields with more than two elements, we now have a shorter proof based upon Jacobson's generalization of Engel's theorem, see [3]).

Recent progress on the topic must be signaled here: in [5], the inequality statement of Theorem 1 has been extended to linear subspaces of $M_n(\mathbb{K})$ with

a *trivial spectrum*, i.e., which consist solely of matrices with no non-zero eigenvalue in \mathbb{K} . The study of such spaces is motivated by its connection with the affine subspaces of matrices with a rank bounded below by some fixed integer. More recently [4], a classification of the linear subspaces of $M_n(\mathbb{K})$ with a trivial spectrum and the maximal dimension $\binom{n}{2}$ has been discovered for fields with more than two elements: for such fields, Theorem 1 appears as an easy consequence of it (see Section 5 of [4]). Finally, in [6], we have been able to prove a theorem similar to Gerstenhaber's for linear subspaces of matrices with exactly one eigenvalue in an algebraic closure of \mathbb{K} .

Both [4] and [6] are based upon a new technique which we will call the *diagonal-compatibility method*. The purpose of this paper is to demonstrate how this strategy can be used to obtain Theorem 2 with essentially no prior knowledge on the topic. In particular, this will yield an alternative proof of Theorem 1 (in the course of the proof, we will point out to some shortcuts for the case $\mathbb{K} = \mathbb{K}_0$). Note that in some cases (e.g., \mathbb{K} is commutative and separable over \mathbb{K}_0), the line of reasoning of [3] may be adapted with some effort by using the trace of \mathbb{K} over \mathbb{K}_0 ; this however fails to yield our more general theorem, so we will not use this strategy.

Our key lemma, which is proven in Section 2, is a variation of Proposition 10 of [5]. It will help us prove both points in Theorem 2: first, point (a) in Section 3 and then point (b) in the longer Section 4.

For to simplify the case $\mathbb{K} = \mathbb{K}_0$, we recall the following classical result, which is proven in [3, 2]. We give a simple proof of it.

Lemma 4. *Assume that \mathbb{K} is commutative, and let A and B be two nilpotent matrices of $M_n(\mathbb{K})$ such that $A + B$ is nilpotent. Then $\text{tr}(AB) = 0$.*

Proof. For $M = (m_{i,j})_{1 \leq i,j \leq n}$, we denote by $c_2(M)$ the coefficient in front of t^{n-2} in the characteristic polynomial of M . Using $c_2(M) = \sum_{1 \leq i < j \leq n} \begin{vmatrix} m_{i,i} & m_{i,j} \\ m_{j,i} & m_{j,j} \end{vmatrix}$, one finds the formula

$$\forall (M, N) \in M_n(\mathbb{K})^2, \quad c_2(M + N) - c_2(M) - c_2(N) = \text{tr}(M) \text{tr}(N) - \text{tr}(MN). \quad (1)$$

As A , B and $A + B$ are nilpotent, we find $\text{tr}(A) = \text{tr}(B) = 0$ and $c_2(A) = c_2(B) = c_2(A + B) = 0$, which yields $\text{tr}(AB) = 0$. \square

2 The key lemma

Definition 1. Let \mathcal{V} be a subset of $M_n(\mathbb{K})$. A vector $X \in \mathbb{K}^n$ is called \mathcal{V} -adapted if it is non-zero and no matrix of \mathcal{V} has $\mathbb{K}X$ as its column space.

Lemma 5. Let \mathcal{V} be a subset of $M_n(\mathbb{K})$ which is closed under addition and contains only nilpotent matrices, and denote by (e_1, \dots, e_n) the canonical basis of \mathbb{K}^n . Then one of the vectors e_1, \dots, e_n is \mathcal{V} -adapted.

The proof is largely similar to that of Proposition 10 in [5].

Proof. The result is trivial for $n = 1$. We use an induction, assuming, given an integer $n \geq 2$, that the result holds for the integer $n - 1$. Let \mathcal{V} be a subset of $M_n(\mathbb{K})$ which is closed under addition and contains only nilpotent matrices. We assume that none of e_1, \dots, e_n is \mathcal{V} -adapted.

For $(i, j) \in \llbracket 1, n \rrbracket^2$, we denote by $E_{i,j}$ the matrix of $M_n(\mathbb{K})$ with a zero entry everywhere except at the (i, j) -spot where the entry is 1. Denote by \mathcal{W} the subset of \mathcal{V} consisting of its matrices with a zero n -th row. Every $M \in \mathcal{W}$ may be written as

$$M = \begin{bmatrix} K(M) & [?]_{(n-1) \times 1} \\ [0]_{1 \times (n-1)} & 0 \end{bmatrix} \quad \text{with } K(M) \in M_{n-1}(\mathbb{K}),$$

so that $K(\mathcal{W})$ consists of nilpotent matrices and is obviously closed under addition. By induction, we know that there is some $i \in \llbracket 1, n-1 \rrbracket$ such that e_i is $K(\mathcal{W})$ -adapted (identifying \mathbb{K}^{n-1} with the subspace $\mathbb{K}^{n-1} \times \{0\}$ of \mathbb{K}^n in the usual way). However, we have assumed that e_i is not \mathcal{V} -adapted, therefore some matrix M of \mathcal{V} has all rows zero except the i -th. Then $M \in \mathcal{W}$, and as e_i is $K(\mathcal{W})$ -adapted, we find that $K(M) = 0$. Thus, $M = a E_{i,n}$ for some $a \in \mathbb{K} \setminus \{0\}$. Now, the same argument may be applied to $P \mathcal{V} P^{-1}$ for any $n \times n$ permutation matrix P . By doing so, we find a map $f : \llbracket 1, n \rrbracket \rightarrow \llbracket 1, n \rrbracket$ and a list $(a_1, \dots, a_n) \in (\mathbb{K} \setminus \{0\})^n$ such that \mathcal{V} contains $a_k E_{f(k),k}$ for all $k \in \llbracket 1, n \rrbracket$. Let us choose a cycle for f , i.e. a list (i_1, \dots, i_p) of pairwise distinct elements of $\llbracket 1, n \rrbracket$ such that $f(i_1) = i_2, \dots, f(i_{p-1}) = i_p$ and $f(i_p) = i_1$. Then, the matrix $M := \sum_{k=1}^p a_{i_k} E_{f(i_k), i_k}$ belongs to \mathcal{V} and satisfies $M^p e_{i_1} = e_{i_1} \left(\prod_{k=1}^p a_{i_{p+1-k}} \right)$. This shows that M is non-nilpotent, which is a contradiction. This *reductio ad absurdum* yields that some e_j is \mathcal{V} -adapted, which concludes the proof by induction. \square

3 Proving the inequality statement

Now, we use Lemma 5 to obtain point (a) of Theorem 2, just as Proposition 10 was used to obtain Theorem 9 in [5].

Again, we use an induction on n . The case $n = 1$ is trivial. Let \mathcal{V} be a nilpotent linear subspace of $M_n(\mathbb{K})$. First of all, we know that some e_i is \mathcal{V} -adapted. Replacing \mathcal{V} with $P\mathcal{V}P^{-1}$ for a well-chosen permutation matrix P , we may assume that e_n is \mathcal{V} -adapted. In that case, we write every matrix of \mathcal{V} as

$$M = \begin{bmatrix} K(M) & C(M) \\ L(M) & a(M) \end{bmatrix},$$

where $K(M)$, $C(M)$, $L(M)$ are respectively $(n-1) \times (n-1)$, $(n-1) \times 1$, $1 \times (n-1)$ matrices, and $a(M) \in \mathbb{K}$. Set

$$\mathcal{W}_1 := \{M \in \mathcal{V} : C(M) = 0\}.$$

Any $M \in \mathcal{W}_1$ is nilpotent, which yields that $a(M) = 0$ and $K(M)$ is nilpotent. Moreover, that e_n is \mathcal{V} -adapted yields:

$$\forall M \in \mathcal{W}_1, K(M) = 0 \Rightarrow M = 0.$$

Using the rank theorem, one finds

$$\dim \mathcal{V} = \dim K(\mathcal{W}_1) + \dim C(\mathcal{V}).$$

As $K(\mathcal{W}_1)$ is a nilpotent linear subspace of $M_n(\mathbb{K})$ and $C(\mathcal{V}) \subset \mathbb{K}^{n-1}$, the induction hypothesis yields

$$\dim \mathcal{V} \leq q \binom{n-1}{2} + q(n-1) = q \binom{n}{2}.$$

Thus, point (a) of Theorem 2 is proven by induction on n .

4 Solving the case of equality

Here, we prove point (b) of Theorem 2 by induction on n . The case $n = 1$ is trivial.

4.1 The case $n = 2$

This case is trivial if $\mathbb{K} = \mathbb{K}_0$ but otherwise needs an explanation. Let A, B be non-zero nilpotent matrices of $M_2(\mathbb{K})$ such that $A + B$ is nilpotent. Assume that $\text{Ker } A \neq \text{Ker } B$. Then $\mathbb{K}^2 = \text{Ker } A \oplus \text{Ker } B$, and we may therefore find a basis (f_1, f_2) of \mathbb{K}^2 such that $f_1 \in \text{Ker } A$ and $f_2 \in \text{Ker } B$. This yields some $P \in \text{GL}_2(\mathbb{K})$ and some $(a, b) \in (\mathbb{K} \setminus \{0\})^2$ such that

$$PAP^{-1} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad PBP^{-1} = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}.$$

Therefore $P(A + B)P^{-1} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$, which is a non-singular matrix. This is a contradiction.

Now, let \mathcal{V} be a q -dimensional linear subspace of the \mathbb{K}_0 -vector space $M_2(\mathbb{K})$ in which every matrix is nilpotent. Choose $A \in \mathcal{V} \setminus \{0\}$. Then we have just shown that every non-zero matrix of \mathcal{V} vanishes on $\text{Ker } A$. Choosing a basis (g_1, g_2) of \mathbb{K}^2 with $g_1 \in \text{Ker } A$, we find a non-singular matrix $P \in \text{GL}_2(\mathbb{K})$ such that every matrix of $P\mathcal{V}P^{-1}$ has a zero first column. As $P\mathcal{V}P^{-1}$ is nilpotent, we deduce that $P\mathcal{V}P^{-1} \subset \text{NT}_2(\mathbb{K})$, and the equality of dimensions yields $P\mathcal{V}P^{-1} = \text{NT}_2(\mathbb{K})$.

4.2 Setting things up for $n \geq 3$

In the rest of the proof, we assume that $n \geq 3$ and that point (b) of Theorem 2 holds for any nilpotent linear subspace of $M_{n-1}(\mathbb{K})$.

Let \mathcal{V} be a nilpotent \mathbb{K}_0 -linear subspace of $M_n(\mathbb{K})$ with dimension $q \binom{n}{2}$. Seing \mathcal{V} as a set of linear endomorphisms of the right- \mathbb{K} -vector space \mathbb{K}^n , what we need is to find a basis (e'_1, \dots, e'_n) of \mathbb{K}^n in which the operators in \mathcal{V} are represented exactly by the strictly upper-triangular $n \times n$ matrices. Our method is to discover such a basis step-by-step. Equivalently, we will replace successively \mathcal{V} with similar linear subspace of matrices in order to simplify \mathcal{V} more and more, until we finally find the space $\text{NT}_n(\mathbb{K})$. Let us quickly lay out the sequence of choices that we will make:

- We will start by choosing the last vector e'_n among the vectors that are \mathcal{V} -adapted. Then we will choose a basis $(\overline{e'_1}, \dots, \overline{e'_{n-1}})$ of the quotient space $\mathbb{K}^n / (e'_n \mathbb{K})$ that is well-suited to \mathcal{V} . Those first two operations will be done within the current section.

- At this point, each one of the vectors e'_1, \dots, e'_{n-1} will be well determined *up to summing a vector of $e'_n \mathbb{K}$* .
- A reasonable choice of e'_2, \dots, e'_{n-1} will then be obtained (Section 4.3).
- A reasonable choice of e'_1 will come last, after a more extensive inquiry (in the end of Section 4.4).

In the rest of the proof, we denote by (e_1, \dots, e_n) the canonical basis of \mathbb{K}^n . As in Section 3, we lose no generality in assuming that e_n is \mathcal{V} -adapted. With the same notation as in Section 3, we deduce from the equality $\dim \mathcal{V} = q \binom{n}{2}$ that

$$\dim K(\mathcal{W}_1) = q \binom{n-1}{2} \quad \text{and} \quad \dim C(\mathcal{V}) = q(n-1).$$

Set

$$\mathcal{V}_{\text{ul}} := K(\mathcal{W}_1)$$

(the subscript “ul” stands for “upper left”). Using the induction hypothesis, we deduce that:

- (A) There exists $Q \in \text{GL}_{n-1}(\mathbb{K})$ such that $Q \mathcal{V}_{\text{ul}} Q^{-1} = \text{NT}_{n-1}(\mathbb{K})$.
- (B) $C(\mathcal{V}) = \mathbb{K}^{n-1}$.

Setting $P_1 := Q \oplus 1$ and replacing \mathcal{V} with $P_1 \mathcal{V} P_1^{-1}$ leaves conditions (A) and (B) unchanged and does not modify the assumption that e_n is adapted to the space under consideration. Therefore, we may now assume, in addition to those properties:

- (A') $\mathcal{V}_{\text{ul}} = \text{NT}_{n-1}(\mathbb{K})$.

4.3 Corner-compatibility and special matrices in \mathcal{V}

Here, we will repeat part of the strategy of Section 4.2. Let $M \in \mathcal{V}$ and assume that M vanishes on e_2, \dots, e_n . Then $M \in \mathcal{W}_1$. Using $K(M) \in \text{NT}_{n-1}(\mathbb{K})$, we find $K(M) = 0$ and therefore $M = 0$. It follows that e_1 is \mathcal{V}^T -adapted.

For any M in \mathcal{V} , we now write:

$$M = \begin{bmatrix} b(M) & R(M) \\ [?]_{(n-1) \times 1} & I(M) \end{bmatrix},$$

where $R(M)$ and $I(M)$ are respectively $1 \times (n-1)$ and $(n-1) \times (n-1)$ matrices, and $b(M) \in \mathbb{K}$. We set

$$\mathcal{W}_2 := \{M \in \mathcal{V} : R(M) = 0\},$$

which is a nilpotent linear subspace of $M_n(\mathbb{K})$. Thus $b(M) = 0$ for every $M \in \mathcal{W}_2$, and $\mathcal{V}_{\text{lr}} := I(\mathcal{W}_2)$ is a nilpotent linear subspace of $M_{n-1}(\mathbb{K})$ (the subscript “lr” stands for “lower-right”). Finally, as e_1 is \mathcal{V}^T -adapted, we find that

$$\forall M \in \mathcal{W}_2, I(M) = 0 \Rightarrow M = 0.$$

Using the rank theorem, we deduce that

$$\dim \mathcal{V} = \dim \mathcal{V}_{\text{lr}} + \dim R(\mathcal{V}).$$

As in Section 4.2, equality $\dim \mathcal{V} = q \binom{n}{2}$ and the induction hypothesis yield:

(C) There exists $Q' \in \text{GL}_{n-1}(\mathbb{K})$ such that $\mathcal{V}_{\text{lr}} = Q' \text{NT}_{n-1}(\mathbb{K}) (Q')^{-1}$.

We aim at modifying \mathcal{V} once more so as to keep (A') and (B) while sharpening (C).

Remark 1. In the rest of the proof, every matrix of $M_n(\mathbb{K})$ will be written as a block matrix with the following shape:

$$\begin{bmatrix} ? & [?]_{1 \times (n-2)} & ? \\ [?]_{(n-2) \times 1} & [?]_{(n-2) \times (n-2)} & [?]_{(n-2) \times 1} \\ ? & [?]_{1 \times (n-2)} & ? \end{bmatrix},$$

where the question marks in the corners represent scalars.

Let us find some special matrices in \mathcal{V} . First of all, (A') yields:

(D) There are \mathbb{K}_0 -linear mappings $\varphi : M_{1,n-2}(\mathbb{K}) \rightarrow M_{1,n-2}(\mathbb{K})$ and $f : M_{1,n-2}(\mathbb{K}) \rightarrow \mathbb{K}$ such that, for every $L \in M_{1,n-2}(\mathbb{K})$, the space \mathcal{V} contains

$$A_L := \begin{bmatrix} 0 & L & 0 \\ 0 & 0 & 0 \\ f(L) & \varphi(L) & 0 \end{bmatrix}.$$

Let $C \in M_{n-2,1}(\mathbb{K})$. By (B), we know that \mathcal{V} contains a matrix of the form $\begin{bmatrix} ? & ? & 0 \\ ? & ? & C \\ ? & ? & ? \end{bmatrix}$. By summing it with a matrix of type A_L , we may assume

furthermore that its first row has the form $[? \ 0 \ \cdots \ 0]$: in that case this row is zero as explained above. Therefore, \mathcal{V} contains a matrix of the following form:

$$\begin{bmatrix} 0 & 0 & 0 \\ ? & ? & C \\ ? & ? & ? \end{bmatrix}. \quad (2)$$

On the other hand, we know from (A') that, for every $U \in \text{NT}_{n-2}(\mathbb{K})$, the subspace \mathcal{V} contains a matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ ? & ? & 0 \end{bmatrix}. \quad (3)$$

We shall now use those observations to prove the following:

Claim 1. *There exists a row matrix $L \in M_{1,n-2}(\mathbb{K})$ such that, for $Q_1 := \begin{bmatrix} I_{n-2} & [0]_{(n-2) \times 1} \\ L & 1 \end{bmatrix}$, one has $Q_1 \mathcal{V}_{\text{lr}} Q_1^{-1} = \text{NT}_{n-1}(\mathbb{K})$.*

Proof. Let us consider a matrix Q' given by property (C). Denote by (e_1, \dots, e_{n-1}) the canonical basis of \mathbb{K}^{n-1} . Then $\mathcal{V}_{\text{lr}} x \subset Q' \text{span}_{\mathbb{K}}(e_1, \dots, e_{n-2})$ for every $x \in \mathbb{K}^{n-1}$. Using the matrices of type (2), we find that $\mathcal{V}_{\text{lr}} e_{n-1}$ contains a $q(n-2)$ -dimensional subspace of the \mathbb{K}_0 -vector space \mathbb{K}^{n-1} . Therefore $\mathcal{V}_{\text{lr}} e_{n-1} = Q' \text{span}_{\mathbb{K}}(e_1, \dots, e_{n-2})$, and in particular $\mathcal{V}_{\text{lr}} e_{n-1}$ is an $(n-1)$ -dimensional \mathbb{K} -linear subspace of \mathbb{K}^{n-1} . Moreover, $\mathcal{V}_{\text{lr}} e_{n-1}$ has a trivial intersection with $e_{n-1} \mathbb{K}$ since every matrix of \mathcal{V} is nilpotent. This yields a \mathbb{K} -linear map $u : \mathbb{K}^{n-2} \rightarrow \mathbb{K}$ such that $\mathcal{V}_{\text{lr}} e_{n-1} = \{(y, u(y)) \mid y \in \mathbb{K}^{n-2}\}$. Writing u as $(y_1, \dots, y_{n-2}) \mapsto a_1 y_1 + \cdots + a_{n-2} y_{n-2}$ for some $(a_1, \dots, a_{n-2}) \in \mathbb{K}^{n-2}$, we set $L := [-a_1 \ \cdots \ -a_{n-2}]$ and $Q_1 := \begin{bmatrix} I_{n-2} & [0]_{(n-2) \times 1} \\ L & 1 \end{bmatrix}$. As $\mathcal{V}_{\text{lr}} x \subset \mathcal{V}_{\text{lr}} e_{n-1}$ for every $x \in \mathbb{K}^{n-1}$, we deduce that the last row of every matrix of $\mathcal{U} := Q_1 \mathcal{V}_{\text{lr}} Q_1^{-1}$ is zero.

We now wish to prove that $\mathcal{U} = \text{NT}_{n-1}(\mathbb{K})$. First of all, any matrix N of $Q_1 \mathcal{V}_{\text{lr}} Q_1^{-1}$ may be written as

$$N = \begin{bmatrix} T(N) & [?]_{(n-2) \times 1} \\ [0]_{1 \times (n-2)} & 0 \end{bmatrix} \quad \text{where } T(N) \text{ is an } (n-2) \times (n-2)\text{-matrix.}$$

Then $T(\mathcal{U})$ is a nilpotent linear subspace of $M_{n-2}(\mathbb{K})$. With the shape of Q_1 and the matrices of type (3), we find that $T(\mathcal{U})$ contains $\text{NT}_{n-2}(\mathbb{K})$. As $\dim T(\mathcal{U}) \leq q \binom{n-2}{2} = \dim \text{NT}_{n-2}(\mathbb{K})$ by point (a) in Theorem 2, we deduce that $T(\mathcal{U}) = \text{NT}_{n-2}(\mathbb{K})$. It follows that $\mathcal{U} \subset \text{NT}_{n-1}(\mathbb{K})$, and the equality of dimensions then yields $\mathcal{U} = \text{NT}_{n-1}(\mathbb{K})$, which finishes the proof. \square

With Q_1 given by Claim 1, we set $P_2 := 1 \oplus Q_1$ and replace \mathcal{V} with $P_2 \mathcal{V} P_2^{-1}$. Then all the preceding properties are unchanged, but we now have the improved:

$$(C') \quad \mathcal{V}_r = \text{NT}_{n-1}(\mathbb{K}).$$

Applying that property to the matrices of type (2) and (3), we find the following properties:

- (E) There is a \mathbb{K}_0 -linear map $h : \text{NT}_{n-2}(\mathbb{K}) \rightarrow \mathbb{K}$ such that, for every $U \in \text{NT}_{n-2}(\mathbb{K})$, the space \mathcal{V} contains the matrix

$$E_U := \begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ h(U) & 0 & 0 \end{bmatrix}.$$

- (F) There are two \mathbb{K}_0 -linear maps $\psi : M_{n-2,1}(\mathbb{K}) \rightarrow M_{n-2,1}(\mathbb{K})$ and $g : M_{n-2,1}(\mathbb{K}) \rightarrow \mathbb{K}$ such that, for every $C \in M_{n-2,1}(\mathbb{K})$, the space \mathcal{V} contains the matrix

$$B_C := \begin{bmatrix} 0 & 0 & 0 \\ \psi(C) & 0 & C \\ g(C) & 0 & 0 \end{bmatrix}.$$

Finally, for every $a \in \mathbb{K}$, property (B) yields that \mathcal{V} contains a matrix with entry a at the $(1, n)$ -spot: combining such a matrix linearly with matrices of type A_L and B_C , we find that \mathcal{V} contains a matrix of the form

$$J_a = \begin{bmatrix} ? & 0 & a \\ ? & ? & 0 \\ ? & ? & ? \end{bmatrix}.$$

4.4 Analyzing φ , ψ , and performing the last change of basis

Claim 2. *For every $L \in M_{1,n-2}(\mathbb{K})$, there exists $a_L \in \mathbb{K}$ such that $\varphi(L) = a_L L$. For every $C \in M_{n-2,1}(\mathbb{K})$, there exists $b_C \in \mathbb{K}$ such that $\psi(C) = C b_C$.*

Proof. Let $(L, C) \in M_{1,n-2}(\mathbb{K}) \times M_{n-2,1}(\mathbb{K})$ be such that $LC = 0$. Setting $M := A_L + B_C$, we compute

$$M^2 = \begin{bmatrix} L\psi(C) & 0 & 0 \\ ? & ? & 0 \\ ? & ? & \varphi(L)C \end{bmatrix}.$$

As $M \in \mathcal{V}$, we know that M^2 is nilpotent and therefore

$$\varphi(L)C = 0 \quad \text{and} \quad L\psi(C) = 0.$$

If we fix $L \in M_{1,n-2}(\mathbb{K})$, varying C yields that the annihilator of the row matrix $\varphi(L)$ contains that of L , and therefore $\varphi(L) = a_L L$ for some $a_L \in \mathbb{K}$. The same line of reasoning yields the second part of Claim 2. \square

Claim 3. *There is a scalar $\lambda \in \mathbb{K}$ such that*

$$\forall (L, C) \in M_{1,n-2}(\mathbb{K}) \times M_{n-2,1}(\mathbb{K}), \quad \varphi(L) = \lambda L \quad \text{and} \quad \psi(C) = -C \lambda.$$

Proof. By Claim 2, there are endomorphisms $\varphi_1, \dots, \varphi_{n-2}$ of the \mathbb{K}_0 -vector space \mathbb{K} such that

$$\forall L = [l_1 \quad \dots \quad l_{n-2}] \in M_{1,n-2}(\mathbb{K}), \quad \varphi(L) = [\varphi_1(l_1) \quad \dots \quad \varphi_{n-2}(l_{n-2})].$$

Applying Claim 2 to the row matrices in which all the entries are equal, we find $\varphi_1 = \dots = \varphi_{n-2}$. As the same line of reasoning applies to ψ , we obtain two endomorphisms u and v of the \mathbb{K}_0 -vector space \mathbb{K} such that

$$\forall L = [l_1 \quad \dots \quad l_{n-2}] \in M_{1,n-2}(\mathbb{K}), \quad \varphi(L) = [u(l_1) \quad \dots \quad u(l_{n-2})]$$

and

$$\forall C = [c_1 \quad \dots \quad c_{n-2}]^T \in M_{n-2,1}(\mathbb{K}), \quad \psi(C) = [v(c_1) \quad \dots \quad v(c_{n-2})]^T.$$

Let $(a, b) \in \mathbb{K}^2$, and set $L_0 := [a \quad 0 \quad \dots \quad 0] \in M_{1,n-2}(\mathbb{K})$ and $C_0 := [b \quad 0 \quad \dots \quad 0]^T \in M_{n-2,1}(\mathbb{K})$. We notice that $M := A_{L_0} + B_{C_0}$ stabilizes the subspace $\text{span}(e_1, e_2, e_n)$ and induces an endomorphism of it represented by $N = \begin{bmatrix} 0 & a & 0 \\ v(b) & 0 & b \\ ? & u(a) & 0 \end{bmatrix}$. Then N is a 3×3 nilpotent matrix, and therefore $N^3 = 0$.

One computes that the entry of N^3 at the $(1, 2)$ -spot is $a(v(b)a + bu(a))$. For $a \neq 0$, this yields

$$v(b)a + bu(a) = 0, \quad (4)$$

which is also obviously true for $a = 0$.

Set now $\lambda := u(1)$. Taking $a = 1$ in (4) yields: $v(b) = -b\lambda$ for all $b \in \mathbb{K}$. Thus, $v(1) = -\lambda$, and taking $b = 1$ in (4) yields $u(a) = \lambda a$ for all $a \in \mathbb{K}$. This finishes the proof of Claim 3. \square

Remark 2. In the case $\mathbb{K} = \mathbb{K}_0$, Claim 3 has a far more simple proof. Indeed, Claim 2 then readily yields a pair $(\lambda, \mu) \in \mathbb{K}^2$ such that $\forall (L, C) \in M_{1, n-2}(\mathbb{K}) \times M_{n-2, 1}(\mathbb{K})$, $\varphi(L) = \lambda L$ and $\psi(C) = \mu C$; as \mathbb{K} is commutative, we find $\text{tr}(A_L B_C) = 0$ for every $(L, C) \in M_{1, n-2}(\mathbb{K}) \times M_{n-2, 1}(\mathbb{K})$, and hence $\mu + \lambda = 0$.

Now, we perform one last change of basis. We set $P := \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_{n-2} & 0 \\ -\lambda & 0 & 1 \end{bmatrix} \in \text{GL}_n(\mathbb{K})$ and we replace \mathcal{V} with $P\mathcal{V}P^{-1}$. Note then that all properties (A'), (B), (C'), (D), (E) and (F) still hold, but we now have a simplified form for the matrices of type A_L and B_C :

$$\forall (L, C) \in M_{1, n-2}(\mathbb{K}) \times M_{n-2, 1}(\mathbb{K}), \quad A_L = \begin{bmatrix} 0 & L & 0 \\ 0 & 0 & 0 \\ f(L) & 0 & 0 \end{bmatrix} \quad \text{and} \quad B_C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & C \\ g(C) & 0 & 0 \end{bmatrix}.$$

From there, our aim is to prove that $\mathcal{V} = \text{NT}_n(\mathbb{K})$. In order to do so, we will show that all the matrices of type A_L , B_C , E_U and J_a are strictly upper-triangular. This will prove the inclusion $\text{NT}_n(\mathbb{K}) \subset \mathcal{V}$, and the equality of dimensions will help us complete the proof. We start by showing that f and g vanish everywhere.

4.5 The vanishing of f and g

Claim 4. *One has $f = 0$ and $g = 0$.*

Proof. We claim that

$$\forall (L, C) \in M_{1, n-2}(\mathbb{K}) \times M_{n-2, 1}(\mathbb{K}), \quad LC \neq 0 \Rightarrow f(L) + g(C) = 0. \quad (5)$$

Let indeed $(L, C) \in M_{1, n-2}(\mathbb{K}) \times M_{n-2, 1}(\mathbb{K})$ be such that $LC \neq 0$; setting $M := A_L + B_C$, we compute $M^3 e_1 = e_1 (f(L) + g(C))LC$ and (5) follows as M^3 is nilpotent.

- Assume that $n \geq 4$. Let $L \in M_{1,n-2}(\mathbb{K})$. As $n - 2 \geq 2$, we may choose $C \in M_{n-2,1}(\mathbb{K}) \setminus \{0\}$ such that $LC = 0$, and then we may choose $L_1 \in M_{1,n-2}(\mathbb{K})$ such that $L_1 C = 1$. Then $(L + L_1)C = 1$, which yields $f(L + L_1) = -g(C) = f(L_1)$. Thus, $f(L) = 0$. The same line of reasoning yields $g = 0$.
- Assume that $n = 3$ and $\#\mathbb{K} > 2$. Let $x \in \mathbb{K}$. Then we may choose $y \in \mathbb{K} \setminus \{0, -x\}$, so that $y \neq 0$ and $x + y \neq 0$. Therefore, $f(x + y) = -g(1) = f(y)$, and hence $f(x) = 0$. The same line of reasoning yields $g = 0$.
- Assume finally that $n = 3$ and $\#\mathbb{K} = 2$, so that $\mathbb{K}_0 = \mathbb{K} \simeq \mathbb{F}_2$. Then, $f(1) = g(1)$. Assume that $f(1) = 1$. Then \mathcal{V} contains the matrices

$$A := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and a matrix of the form

$$J = \begin{bmatrix} a & 0 & 1 \\ b & c & 0 \\ d & e & f \end{bmatrix}.$$

Note that \mathbb{K} is commutative, thus Lemma 4 yields $\text{tr}(AJ) = \text{tr}(BJ) = 0$, and hence $b = e = 1$. As J is nilpotent, we also have $\text{tr}(J) = 0$, and hence $f = a + c$. Using $\forall t \in \mathbb{K}, t^2 = t$ and $2t = 0$, we finally compute:

$$\forall (x, y) \in \mathbb{K}^2, 0 = \det(J + xA + yB) = 1 + cd + (a + c)y + ax + dxy.$$

This yields both $cd = 1$ and $d = 0$, a contradiction.

Therefore, $f(1) = g(1) = 0$, and so $f = 0$ and $g = 0$, as claimed.

□

4.6 The vanishing of h

Claim 5. *One has $h = 0$.*

Proof. Let $U \in \text{NT}_{n-2}(\mathbb{K})$ be such that $U^2 = 0$. Set $L_0 := \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \in \text{M}_{1,n-2}(\mathbb{K})$ and $C_0 := L_0^T$, so that $L_0 U C_0 = 0$ and $L_0 C_0 = 1$. Setting $M := A_{L_0} + B_{C_0} + E_U$, one checks that $M^3 e_n = e_n h(U)$, and therefore $h(U) = 0$. In particular, $h(E_{i,j} a) = 0$ for every $a \in \mathbb{K}$ and every $(i, j) \in \llbracket 1, n \rrbracket^2$ with $j > i$ (where $E_{i,j}$ is the matrix with all entries zero except at the (i, j) -spot where the entry is 1). As h is additive, we deduce that h vanishes everywhere on $\text{NT}_{n-2}(\mathbb{K})$. \square

4.7 The matrices of type J_a

4.7.1 Simplifying the J_a matrices

Let us sum up. For every triple $(L, C, U) \in \text{M}_{1,n-2}(\mathbb{K}) \times \text{M}_{n-2,1}(\mathbb{K}) \times \text{NT}_{n-2}(\mathbb{K})$, the space \mathcal{V} contains the matrices

$$A_L = \begin{bmatrix} 0 & L & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & C \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E_U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Adding an appropriate E_U to each matrix of type J_a , one finds \mathbb{K}_0 -linear maps $\alpha : \mathbb{K} \rightarrow \mathbb{K}$, $\beta : \mathbb{K} \rightarrow \mathbb{K}$, $\gamma : \mathbb{K} \rightarrow \mathbb{K}$, $L_1 : \mathbb{K} \rightarrow \text{M}_{1,n-2}(\mathbb{K})$, $C_1 : \mathbb{K} \rightarrow \text{M}_{n-2,1}(\mathbb{K})$, $T : \mathbb{K} \rightarrow \text{LT}_{n-2}(\mathbb{K})$ (where $\text{LT}_{n-2}(\mathbb{K})$ denotes the set of lower-triangular matrices of $\text{M}_{n-2}(\mathbb{K})$) such that, for every $a \in \mathbb{K}$, the subspace \mathcal{V} contains

$$J_a := \begin{bmatrix} \alpha(a) & 0 & a \\ C_1(a) & T(a) & 0 \\ \beta(a) & L_1(a) & \gamma(a) \end{bmatrix}.$$

Our aim in what follows is to prove:

Claim 6. *All the maps α , β , γ , L_1 , C_1 and T vanish everywhere on \mathbb{K} .*

We have to distinguish between two cases, the main problem being the handling of fields with two elements.

4.7.2 Proof of Claim 6: the case $\mathbb{K} = \mathbb{K}_0$

We assume $\mathbb{K} = \mathbb{K}_0$. In particular, \mathbb{K} is commutative, which allows us to use Lemma 4 to obtain $\text{tr}(J_1 A_L) = 0$, $\text{tr}(J_1 B_C) = 0$ and $\text{tr}(J_1 E_U) = 0$ for all $(L, C, U) \in \text{M}_{1,n-2}(\mathbb{K}) \times \text{M}_{n-2,1}(\mathbb{K}) \times \text{NT}_{n-2}(\mathbb{K})$. Therefore, $L_1(1) = 0$, $C_1(1) = 0$ and $T(1)$ is a diagonal matrix. Every diagonal entry of $T(1)$ is an eigenvalue of

J_1 , and hence $T(1) = 0$. Then J_1 induces an endomorphism of $\text{span}(e_1, e_n)$ whose matrix in (e_1, e_n) is $N = \begin{bmatrix} \alpha(1) & 1 \\ \beta(1) & \gamma(1) \end{bmatrix}$. This last matrix must be nilpotent, and hence $\alpha(1) = -\gamma(1)$ and $\beta(1) = -\gamma(1)^2$ (as $\text{tr } N = 0$ and $\det N = 0$). Choose finally $(L, C) \in M_{1, n-2}(\mathbb{K}) \times M_{n-2, 1}(\mathbb{K})$ such that $LC \neq 0$, and set $M := J_1 + A_L + B_C$. One checks that $M^3 e_1 = -\gamma(1)^2 LC e_1$, and hence $\gamma(1) = 0$. Therefore, the maps $\alpha, \beta, \gamma, L_1, C_1$ and T all vanish on 1; since they are \mathbb{K} -linear, Claim 6 is proven in the case $\mathbb{K} = \mathbb{K}_0$.

4.7.3 Proof of Claim 6: the case $\#\mathbb{K} > 2$

We assume here that $\#\mathbb{K} > 2$, which holds whenever $\mathbb{K}_0 \subsetneq \mathbb{K}$.

Fix $a \in \mathbb{K}$. Let $C_0 \in M_{n-2, 1}(\mathbb{K}) \setminus \{0\}$. Let $x \in \mathbb{K}$. We consider the non-zero vector $X := \begin{bmatrix} x \\ C_0 \\ 1 \end{bmatrix}$ of \mathbb{K}^n . The \mathbb{K}_0 -vector space $\mathcal{V}X$ must intersect $X \mathbb{K}$ trivially as all the elements of \mathcal{V} are nilpotent. Thus $\dim \mathcal{V}X \leq (n-1)q$. However, for every $(L, C) \in M_{1, n-2}(\mathbb{K}) \times M_{n-2, 1}(\mathbb{K})$, we have

$$A_L X = \begin{bmatrix} LC_0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad B_C X = \begin{bmatrix} 0 \\ C \\ 0 \end{bmatrix}$$

Varying L and C then yields the inclusion $\mathbb{K}^{n-1} \times \{0\} \subset \mathcal{V}X$. Since $\dim \mathcal{V}X \leq (n-1)q = \dim(\mathbb{K}^{n-1} \times \{0\})$, we deduce that $\mathcal{V}X = \mathbb{K}^{n-1} \times \{0\}$. However, the last entry of $J_a X$ is $\beta(a)x + L_1(a)C_0 + \gamma(a)$, and therefore:

$$\forall x \in \mathbb{K}, \quad \beta(a)x + L_1(a)C_0 + \gamma(a) = 0.$$

We deduce that $L_1(a)C_0 + \gamma(a) = 0$ and $\beta(a) = 0$, which yields:

$$\forall C \in M_{n-2, 1}(\mathbb{K}) \setminus \{0\}, \quad \forall y \in \mathbb{K} \setminus \{0\}, \quad L_1(a)Cy + \gamma(a) = 0.$$

As $\#\mathbb{K} > 2$, we deduce that $\gamma(a) = 0$ and

$$\forall C \in M_{n-2, 1}(\mathbb{K}) \setminus \{0\}, \quad L_1(a)C = 0.$$

Varying C then yields $L_1(a) = 0$.

Let again $C_0 \in M_{1,n-2}(\mathbb{K}) \setminus \{0\}$, and set $Y := \begin{bmatrix} 1 \\ C_0 \\ 0 \end{bmatrix}$. For every $(L, C) \in M_{1,n-2}(\mathbb{K}) \times M_{n-2,1}(\mathbb{K})$, we have

$$A_L^T Y = \begin{bmatrix} 0 \\ L^T \\ 0 \end{bmatrix} \quad \text{and} \quad B_C^T X = \begin{bmatrix} 0 \\ 0 \\ C^T C_0 \end{bmatrix}.$$

As above, varying C and L yields $\mathcal{V}^T Y = \{0\} \times \mathbb{K}^{n-1}$. The first entry of $J_a^T Y$ is $\alpha(a) + C_0^T C_1(a)$ and it must be 0. Again, varying C_0 yields both $\alpha(a) = 0$ and $C_1(a) = 0$.

Let $U \in \text{NT}_{n-2}(\mathbb{K})$. For every $t \in \mathbb{K}_0$, the matrix $E_U + tJ_a$ is nilpotent and stabilizes $\text{span}(e_2, \dots, e_{n-1})$, with an induced endomorphism represented in (e_2, \dots, e_{n-1}) by $U + tT(a)$. It follows that $\text{NT}_{n-2}(\mathbb{K}) + \mathbb{K}_0 T(a)$ is a nilpotent linear subspace of $M_{n-2}(\mathbb{K})$. If $T(a) \neq 0$, then we have a contradiction with point (a) of Theorem 2. Therefore $T(a) = 0$, and Claim 6 is proven.

4.8 Conclusion

We have shown that, for every list $(L, C, U, a) \in M_{1,n-2}(\mathbb{K}) \times M_{n-2,1}(\mathbb{K}) \times \text{NT}_{n-2}(\mathbb{K}) \times \mathbb{K}$, the additive group \mathcal{V} contains all four matrices

$$\begin{bmatrix} 0 & L & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & C \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows that \mathcal{V} contains $\text{NT}_n(\mathbb{K})$. As $\dim \mathcal{V} = q \binom{n}{2} = \dim \text{NT}_n(\mathbb{K})$, we conclude that $\mathcal{V} = \text{NT}_n(\mathbb{K})$. This completes our proof of point (b) of Theorem 2.

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